

Homotopy classification of finite group actions on aspherical spaces

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Abstract

The author proposes a method for investigating actions of finite groups on aspherical spaces. Complete homotopy classification of free actions of finite groups on aspherical spaces is obtained. Also there are some results about non-free actions. For example a relation between the cohomology of finite groups and the lattice structure of its subgroups is obtained by the proposed method. This relation is formulated in terms of spectral sequences.

1 Introduction

Suppose G is a finite group, T is a Hausdorff topological space, and $p : G \rightarrow \text{Homeo } T$ is a free action of G on T . Therefore the natural projection $P : T \rightarrow T/p(G)$ is a regular covering. Now suppose that T is an Eilenberg-MacLane space of type $K(\pi, 1)$; then the long exact sequence of the covering has the form

$$1 \rightarrow \pi \rightarrow \pi_1(T/p(G)) \rightarrow G \rightarrow 1. \quad (1)$$

Thus we have an extension of G by π . The aim of this paper is to prove that the set of all free actions (up to homotopy conjugation) of a finite group G on aspherical spaces of type $K(\pi, 1)$ is in one-to-one correspondence with the set of all extensions of the group G by the group π ¹. Classification of such extensions is well known (see [5]) and closely connected to the cohomology of the group G .

Also some results about non-free actions are obtained in this paper. For example if G acts (not necessarily freely) on T then the cohomology of the group G and the space T are related by a Hochschild-Mostov spectral sequence. In particular a relation between cohomology of a finite group and the subgroup lattice structure is detected.

¹This statement is proved under minimal natural restriction: existence of cell complex structure is used

2 Free actions on aspherical spaces

Let T be an Eilenberg-MacLane space of type $K(\pi, 1)$, i.e., $\pi_k(T) = 0$ for all $k \neq 1$ and the groups $\pi_1(T)$ and π are isomorphic. Sometimes we will use the term “aspherical space” instead of “Eilenberg-MacLane space of type $K(\pi, 1)$ ”.

Remark. We will work only with Hausdorff spaces. Throughout what follows the term “space” will mean “Hausdorff topological space”.

Also let G be a finite group.

Definition 2.1. Let p_1 and p_2 be actions of finite groups G_1 and G_2 on spaces T_1 and T_2 . The action p_1 is called homotopy conjugate to p_2 ($p_1 \sim p_2$) if there exist a homotopy equivalence $\varphi : G_1 \rightarrow G_2$ and an isomorphism $\theta : G_1 \rightarrow G_2$ such that

$$\varphi \circ p_1(g) = p_2(\theta(g)) \circ \varphi \text{ for all } g \in G_1. \quad (2)$$

Remark. Notice that the classification problem of finite group actions (up to homotopy conjugation) is not well defined in general. It is clear that the relation \sim is reflexive and transitive but it is not symmetric. For example suppose \mathbb{Z}_2 acts on infinite dimensional sphere S^∞ by two ways: trivially $p_1(\pm 1)x = x$ and freely $p_2(\pm 1)x = \pm x$; then $p_2 \sim p_1$ by homotopy equivalence $\varphi : S^\infty \rightarrow S^\infty$ such that $\varphi : S^\infty \mapsto x_0 \in S^\infty$ but there is no map $\varphi' : S^\infty \rightarrow S^\infty$ such that $\varphi' \circ p_1(\pm 1) = p_2(\pm 1) \circ \varphi'$. However this section is devoted to free actions on aspherical spaces and we will prove that the relation \sim is symmetric in this particular case.

We will need the following minimal natural restriction for the consequent proofs:

Definition 2.2. A free action p of a finite group G on a space T is called regular if the quotient space $T/p(G)$ is a cell complex. Note that if p is a regular free action on a space T then T is also a cell complex.

For example if G acts on an polyhedron ² T freely and simplicially then p is regular. Of course there are many other examples of regular free actions.

Our aim is to show that the set of all regular free actions on an aspherical space (up to homotopy conjugation) is in one-to-one correspondence with the set of all classes of equivalent extensions of the form (2).

Definition 2.3. Extensions $1 \rightarrow \pi_i \rightarrow S_i \rightarrow G_i \rightarrow 1$, $i = 1, 2$ are called equivalent if they are isomorphic in category of short exact sequences, i.e., there exist isomorphisms $\rho : \pi_1 \rightarrow \pi_2$, $\xi : S_1 \rightarrow S_2$, and $\theta : G_1 \rightarrow G_2$ such that the following diagram is commutative

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1 & \longrightarrow & S_1 & \longrightarrow & G_1 \longrightarrow 1 \\ & & \downarrow \rho & & \downarrow \xi & & \downarrow \theta \\ 1 & \longrightarrow & \pi_2 & \longrightarrow & S_2 & \longrightarrow & G_2 \longrightarrow 1 \end{array}$$

²A polyhedron is a space homeomorphic to the geometric realization of an simplicial complex with weak topology.

Lemma 2.1 (well-definiteness). *Let p_1 and p_2 be free actions of finite groups G_1 and G_2 on path-connected (not necessarily aspherical) spaces T_1 and T_2 . Suppose there exist a homotopy equivalence $\varphi : T_1 \rightarrow T_2$ and an isomorphism $\theta : G_1 \rightarrow G_2$ such that the equation (2) are satisfied; then the short exact sequences*

$$1 \rightarrow \pi_1(T_i) \rightarrow \pi_1(T_i/p_i(G_i)) \rightarrow G_i \rightarrow 1, \quad i = 1, 2$$

are equivalent (these sequences are induced by the coverings $T_i \rightarrow T_i/p_i(G_i)$).

Throughout what follows by $f_\#$ denote the induced homomorphism $f_\# : \pi_k(T_1) \rightarrow \pi_k(T_2)$ of homotopy groups where $f : T_1 \rightarrow T_2$ is a continuous map.

Proof. Let us mark a point x_i^0 in the space T_i and $y_i^0 = P_i(x_i^0)$ in the space $T_i/p_i(G_i)$ where $P_i : T_i \rightarrow T_i/p_i(G_i)$ is the natural projection, $i = 1, 2$. Assume that $x_2^0 = \varphi(x_1^0)$.

Let $\psi : T_1/p_1(G_1) \rightarrow T_2/p_2(G_2)$ be the map induced by the map φ . Since projection P_1 is open, we see that the map ψ is continuous. By construction we see that the following diagram is commutative

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(T_1, x_1^0) & \longrightarrow & \pi_1(T_1/p_1(G_1), y_1^0) & \longrightarrow & G_1 \longrightarrow 1 \\ & & \downarrow \varphi_\# & & \downarrow \psi_\# & & \downarrow \theta \\ 1 & \longrightarrow & \pi_1(T_2, x_2^0) & \longrightarrow & \pi_1(T_2/p_2(G_2), y_2^0) & \longrightarrow & G_2 \longrightarrow 1 \end{array}$$

It is clear that $\varphi_\#$ is an isomorphism. Using 5-lemma for the diagram we see that the homomorphism $\psi_\#$ is an isomorphism too. \square

Corollary 2.1. *If the free action p_1 is homotopy conjugate to p_2 then the quotient spaces $T_1/p_1(G_1)$ and $T_2/p_2(G_2)$ are weakly homotopy equivalent. Moreover, if the spaces $T_1/p_1(G_1)$ and $T_2/p_2(G_2)$ are cell complexes then from the Whitehead theorem it follows that these spaces are homotopy equivalent.*

Lemma 2.2 (injectiveness). *Let p_1 and p_2 be regular free actions of finite groups G_1 and G_2 on aspherical spaces T_1 and T_2 . Suppose there exist isomorphisms $\xi : \pi_1(T_1/p_1(G_1)) \rightarrow \pi_1(T_2/p_2(G_2))$ and $\theta : G_1 \rightarrow G_2$ such that the following square is commutative (horizontal arrows are the boundary maps)*

$$\begin{array}{ccc} \pi_1(T_1/p_1(G_1)) & \xrightarrow{\partial_1} & G_1 \\ \downarrow \xi & & \downarrow \theta \\ \pi_1(T_2/p_2(G_2)) & \xrightarrow{\partial_2} & G_2 \end{array}$$

Then the action p_1 is homotopy conjugate to p_2 by the isomorphism θ and a homotopy equivalence $\varphi : T_1 \rightarrow T_2$ such that $\xi \circ P_{1\#} = P_{2\#} \circ \varphi_\#$.

Proof. Mark points $x_1^0 \in T_1^0$ and $y_1^0 = P_1(x_1^0) \in T_1/p_1(G_1)$. By definition, put $y_2^0 = \psi(y_1^0)$. Denote by x_2^0 any point belonging to the discrete preimage $P_2^{-1}(y_2^0)$.

Since ξ is isomorphism and CW-complexes $T_i/p_i(G_i)$, $i = 1, 2$ are aspheric we see that there exists homotopy equivalence $\psi : T_1/p_1(G_1) \rightarrow T_2/p_2(G_2)$ such that $\psi(y_1^0) = y_2^0$ and $\psi_\# = \xi$.

Let us construct the required map $\varphi : T_1 \rightarrow T_2$: for any point $x \in T_1$ there exists a path $\gamma : [0; 1] \rightarrow T_1$ joining $x_1^0 = \gamma(0)$ to $x = \gamma(1)$. Consider the path $\psi \circ P_1 \circ \gamma$ in the space $T_2/p_2(G_2)$ joining y_2^0 to $\psi(P_1(x))$. There exists a unique lifting γ' of this path to the space T_2 such that $\gamma'(0) = x_2^0$. So $P_2 \circ \gamma' = \psi \circ P_1 \circ \gamma$. By definition, put $\varphi(x) = \gamma'(1)$.

First we shall prove that the map φ is well defined. Obviously, it is sufficient to prove that if γ is a loop in T_1 ($x = x_1^0$) then $\varphi(x_1^0) = x_2^0$. Let $[\gamma] \in \pi_1(T_1, x_1^0)$ be the homotopy class of the loop γ ; then $P_{1\#}([\gamma])$ belongs to the kernel $\ker \partial_1$. Since the diagram from the condition is commutative, we have $\psi_\#(P_{1\#}([\gamma])) \in \ker \partial_2$. Since covering $T_2 \rightarrow T_2/p_2(G_2)$ is regular and $P_2 \circ \gamma' = \psi \circ P_1 \circ \gamma$, we see that γ' is a loop in T_2 as was to be proved.

Notice that the following diagram is commutative by construction of φ :

$$\begin{array}{ccc} T_1 & \xrightarrow{\varphi} & T_2 \\ \downarrow P_1 & & \downarrow P_2 \\ T_1/p_1(G_1) & \xrightarrow{\psi} & T_2/p_2(G_2) \end{array}$$

It is important to note that if γ is a path in T_2 then the path $\varphi \circ \gamma$ in T_2 is the same as the lift γ' of the path $\psi \circ P_1 \circ \gamma$ into the space T_2 such that $\gamma'(0) = \varphi(\gamma(0))$.

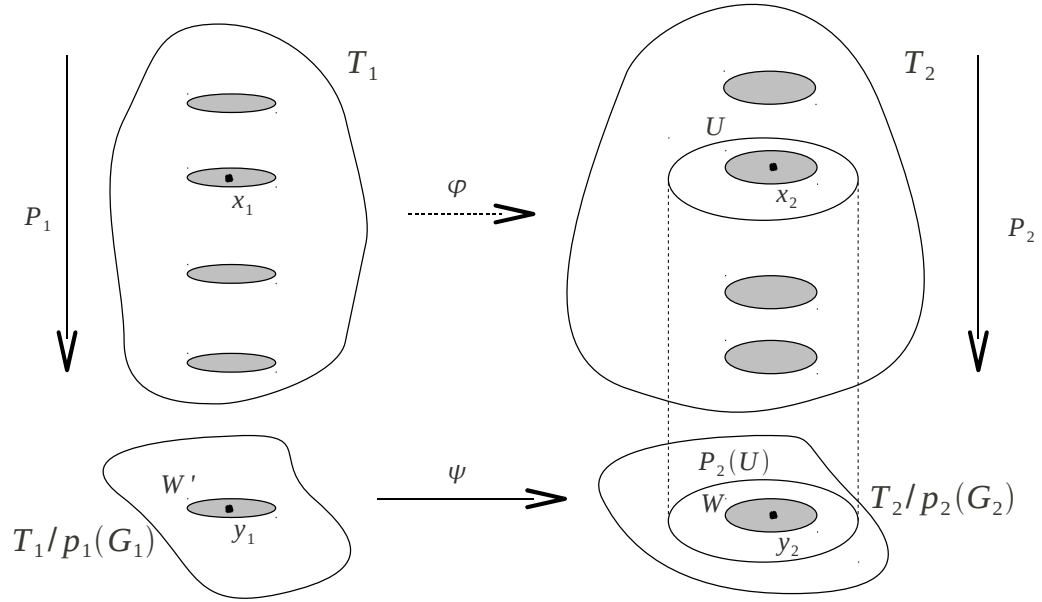
Secondly we shall show that the constructed map φ is continuous. Let x_1 be a point in T_1 , $x_2 = \varphi(x_1) \in T_2$, U a neighborhood of the point x_2 in T_2 , and $y_i = P_i(x_i)$ where $i = 1, 2$. It is sufficient to find a neighborhood of the point x_1 such that $\varphi(V) \subseteq U$.

Since the projection P_2 is open, we see that $P_2(U)$ is open too. Hence there exists a neighborhood $W \subseteq P_2(U)$ of the point y_2 such that $P_2^{-1}(W)$ is homeomorphic to the direct product $P_2^{-1}(y_2) \times W$. The set $x_2 \times W$ is open in T_2 because the fibre $P_2^{-1}(y_2)$ is discrete.

The sets $(x_2 \times W) \cap U$ and $P_2((x_2 \times W) \cap U)$ are open. Whence without loss of generality it can be assumed that $x_2 \times W$ is contained in U (we can replace W by $P_2((x_2 \times W) \cap U)$ if necessary).

It follows from continuity of ψ that $\psi^{-1}(W)$ is open in $T_1/p_1(G)$. Consequently there exists a neighborhood $W' \subseteq \psi^{-1}(W)$ of the point y_2 such that the preimage $P_1^{-1}(W')$ is homeomorphic to $P_1^{-1}(y_1) \times W'$. Since the cell complex $T_1/p_1(G_1)$ is locally contractible (see [8]), we may assume that the neighborhoods W' and $x_1 \times W'$ are path-connected.

By $P_2(\varphi(x_1 \times W')) = \psi(W') \subseteq W$ so we see that $\varphi(x_1 \times W')$ is locally connected too. Therefore, we see that $\varphi(x_1 \times W')$ belongs to the level $x_2 \times W \subseteq U$. This means that φ is continuous as was to be proved.



Thirdly let us prove that φ is a homotopy equivalence of T_1 and T_2 . From the Whitehead theorem it follows that φ is a homotopy equivalence iff $\varphi_{\#}$ is an isomorphism of the groups $\pi_1(T_1)$ and T_2 . The following diagram is commutative:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \pi_1(T_1) & \xrightarrow{P_{1\#}} & \pi_1(T_1/p_1(G_1)) & \longrightarrow & G_1 \\
 & & \downarrow \varphi_{\#} & & \downarrow \psi_{\#} & & \downarrow \theta \\
 1 & \longrightarrow & \pi_1(T_2) & \xrightarrow{P_{2\#}} & \pi_1(T_2/p_2(G_2)) & \longrightarrow & G_2
 \end{array}$$

Using 5-lemma we see that $\varphi_{\#}$ is an isomorphism.

Finally we shall prove that the map φ is equivariant³. For any $g \in G_1$ there exists a path γ_g connecting points x_1^0 and $p_1(g)x_1^0$. Let x be a point in T_1 , γ be a path connecting x_1^0 and x and $\omega = \gamma_g \oplus (p_1(g) \circ \gamma)$ be the path connecting x_1^0 and $p_1(g)x$ (by definition, $\omega(0) = \gamma_g(0) = x_1^0$, $\omega(\frac{1}{2}) = \gamma_g(1) = p_1(g)x_1^0$ and $\omega(1) = p_1(g)\gamma(1) = p_1(g)x$).

Let us construct an explicit lift of the path $\psi \circ P_1 \circ \omega$ into the space T_2 . Note that since $\psi \circ P_1 \circ \gamma_g$ is a loop, we get $P_2(\varphi(p_1(g)x_1^0)) = y_2^0$. Hence there exists an element $g' \in G_2$ such that $\varphi(p_1(g)x_1^0) = p_2(g')x_2^0$. We obtain $g' = \theta(g)$ by commutativity of the diagram from the condition.

Consider the lift γ'_g of the path $\psi \circ P_1 \circ \gamma_g$ such that $\gamma'_g(0) = x_2^0$. By the above we get $\gamma'_g(1) = p_2(\theta(g))x_2^0$. Thus if γ' is the lift of the path $\psi \circ P_1 \circ \gamma$

³The term “equivariant” is usually used for actions of one group. Here we mean “equivariant with the respect to the isomorphism θ ”.

such that $\gamma'(0) = x_2^0$; then the path $\omega' = \gamma'_g \oplus (p_2(\theta(g)) \circ \gamma')$ is the lift of the path $\psi \circ P_1 \circ \omega$ such that $\omega'(0) = x_2^0$. Consequently, $\varphi(p_1(g)x) = \omega'(1) = p_2(\theta(g))\gamma'(1) = p_2(\theta(g))x$. Q.E.D. \square

Remark. If the action p_1 is homotopy conjugate to p_2 then the spaces T_1 and T_2 are homotopy equivalent.

Corollary 2.2 (symmetry of \sim). *If free actions p_1 and p_2 are regular then $p_1 \sim p_2$ iff $p_2 \sim p_1$.*

Proof. Suppose, $p_1 \sim p_2$. Using lemma 2.1 we see that there exists an isomorphism $\xi : \pi_1(T_1/p_1(G)) \rightarrow \pi_1(T_2/p_2(G))$ such that the appropriate diagram is commutative. Using the previous lemma to the isomorphisms ξ^{-1} and θ^{-1} we immediately get $p_2 \sim p_1$. \square

Lemma 2.3 (surjectiveness). *Let $1 \rightarrow \pi \rightarrow S \rightarrow G \rightarrow 1$ be an extension of a finite group G by a discrete group π ; then there exist an aspheric space T of type $K(\pi, 1)$ and a regular action p of G on T such that the following short sequence is equivalent to the first one:*

$$1 \rightarrow \pi_1(T) \rightarrow \pi_1(T/p(G)) \rightarrow G \rightarrow 1$$

Proof. Let us use the classic the Milnor construction (see [3, 4]). Consider the group S as a discrete topological space; then the space $\mathbf{E}S = S * S * S * \dots$ is a join of countable number of the S copies. The space $\mathbf{E}S$ consists of points

$$\langle t, x \rangle = (t_1 x_1, t_2 x_2, \dots, t_n x_n, \dots),$$

where $x_i \in S$ and $t_i \in [0; 1]$. Moreover, only finite number of t_i is non-zero. Points $\langle t, x \rangle$ and $\langle t', x' \rangle$ are considered identical if $t_i = t'_i$ for all i and $x_i = x'_i$ when $t_i = t'_i > 0$. By definition the topology on $\mathbf{E}S$ is the weakest topology such that the following maps are continuous:

$$\begin{aligned} T_i : \mathbf{E}G &\rightarrow [0; 1]; & T_i(\langle t, x \rangle) &= t_i; \\ X_i : T_i^{-1}(0; 1] &\rightarrow S; & X_i(\langle t, x \rangle) &= x_i; \end{aligned}$$

It is obvious that the space $\mathbf{E}S$ is an simplicial complex.

The left action of S on $\mathbf{E}S$ is free, continuous and simplicial. We claim that this action is discrete, i.e. for any point $\langle t, x \rangle$ there exists a neighborhood $U \ni \langle t, x \rangle$ such that the sets gU , $g \in G$ are mutually disjoint. Indeed,

$$U = \bigcap_{i: t_i \neq 0} (T_i^{-1}(0; 1] \cap X_i^{-1}(x_i)).$$

Since U is an intersection of finite number of open sets; then U is open. It is obvious that the sets gU are mutually disjoint.

Thus the group π as an subgroup of S acts freely and discretely on $\mathbf{E}S$. Consequently $\mathbf{E}S \rightarrow T = \mathbf{E}S/\pi$ is a regular covering. Obviously T is an Eilenberg-MacLane space of type $K(\pi, 1)$. Since action of π on $\mathbf{E}S$ is simplicial, we see that T has an simplicial structure (see [1]).

Clearly there exists a continuous and free action of $G = S/\pi$ on T and $T/G = \mathbf{E}S/S$. \square

Finally let us note that we can not weaken the conditions of the previous lemma, i.e., we can not require that for any aspheric space T there exists a necessary free action of the group G . For example let T be a wedge product of two circles. Since any homeomorphism leaves fixed the sewing point, we see that for any non-trivial group there is no its free action on T .

Theorem 2.1 (classification). *Let G be a finite group and π a discrete group. Then the set of all regular free actions (up to homotopy conjugation) of G on Eilenberg-MacLane spaces of type $K(\pi, 1)$ is in one-to-one correspondence with the set of all classes of equivalent extensions of G by π . Here if p is a regular free action on an aspheric space T then the following extension corresponds to the action p :*

$$1 \rightarrow \pi_1(T) \rightarrow \pi_1(T/p(G)) \rightarrow G \rightarrow 1.$$

Let us consider a simple illustrative example:

Example. Let the group $G = \mathbb{Z}_2$ act on $T = S^1 \times S^\infty$. Since S^∞ is contractible, then T is an aspherical space of type $K(\mathbb{Z}, 1)$. There exists only three different extensions of the group \mathbb{Z}_2 by \mathbb{Z} :

1. The group \mathbb{Z}_2 acts on S^1 trivially. So, $p_1(\pm 1)(x, y) = (x, \pm y)$ where $(x, y) \in S^1 \times S^\infty$. Then we get $(S^1 \times S^\infty)/p_1(\mathbb{Z}_2) = S^1 \times \mathbb{R}P^\infty$. Thus we have the first extension:

$$1 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow 1.$$

2. The group \mathbb{Z}_2 acts on S^1 by reflection with respect to the center. So, $p_2(\pm 1)(x, y) = (\pm x, y)$. Since $(S^1 \times S^\infty)/p_2(\mathbb{Z}_2) = S^1 \times S^\infty$, then we obtain the second extension

$$1 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \rightarrow \mathbb{Z}_2 \rightarrow 1.$$

3. The group \mathbb{Z}_2 acts on $S^1 \subseteq \mathbb{C}$ by reflection with respect to a diameter. So, $p_3(\pm 1)(x, y) = (x^{\pm 1}, \pm y)$. Hence we get the last one

$$1 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rtimes_{(-1)} \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow 1.$$

Here we have to a little bit look ahead. Now we have no effective methods for calculating of this extension. But we will prove lemma 5.1 and the extension can be easily calculated by this lemma.

3 Congruent extensions

Classification of classes of congruent extensions is well known (Eulenberg, MacLane, 1947, see [6]).

Definition 3.1. Equivalent extensions $1 \rightarrow \pi_i \rightarrow S_i \rightarrow G_i \rightarrow 1$, $i = 1, 2$ are called congruent if $\pi_1 = \pi_2 = \pi$, $G_1 = G_2 = G$ and $\rho = id_\pi$, $\theta = id_G$

$$\begin{array}{ccccccc} & & & S_1 & & & \\ & \nearrow & & \downarrow \xi & \searrow & & \\ 1 & \longrightarrow & \pi & & G & \longrightarrow & 1 \\ & \searrow & & \downarrow & \nearrow & & \\ & & & S_2 & & & \end{array}$$

We can easy reformulate constructed homotopy classification in theorem 2.1 for the congruent extensions language:

Definition 3.2. An aspheric space T is called an Eilenberg-MacLane space of type $K(\pi, 1)$ in strong sense if it is marked by a point $x^0 \in T$ and an isomorphism $\rho : \pi_1(T, x^0) \rightarrow \pi$ is fixed.

Since the isomorphism ρ is fixed, we can use π instead of $\pi_1(T, x^0)$.

Theorem 3.1 (classification). *Let G be a finite group and π a discrete group. Then the set of all regular free actions (up to strong homotopy conjugation) of G on Eilenberg-MacLane spaces of type $K(\pi, 1)$ in strong sense is in one-to-one correspondence with the set of all classes of congruent extensions of G by π . Here if p is a regular free action of G on an aspheric space T of type $K(\pi, 1)$ in strong sense then the following extension corresponds to the action p :*

$$1 \rightarrow \pi_1(T) \rightarrow \pi_1(T/p(G)) \rightarrow G \rightarrow 1.$$

A proof of this theorem is similar to the proof of the theorem 2.1 and based on lemmas almost word for word repeated the lemmas on well-definiteness, injectiveness and surjectiveness.

Example. If the group π is abelian then the action p of G on T induces the action η of G on π ($\eta : G \rightarrow \text{Aut } \pi$). G -module structure on π is defined by the map η . The set of all congruence classes of extensions of G by π is exactly the second cohomology $H^2(G, \pi)$ (the structure of G -module on π is fixed). If G is non-abelian then there exists only homomorphism $\eta : G \rightarrow \text{Out } \pi$ of G into the outer automorphism class group of the group π . In this case the set of all congruence classes of extensions of G by π is the second cohomology $H^2(G, C(\pi))$ ($C(\pi)$ is the center of π) or empty. This depends on an obstruction in the third cohomology $H^3(G, C(\pi))$.

The results of the classification theorems 2.1 and 3.1 can be generalize to infinite discrete groups. However in this case the projection $T \rightarrow T/G$ may not be a covering. Since we essentially use the covering structure on $T \rightarrow T/G$, then one have to require essentially stronger conditions than we used.

4 Non-free actions

Let us remember the S^∞ example from the first paragraph. We see that there are some difficulties with the notion of homotopy conjugation for non-free actions. So let us use the Borel construction.

Definition 4.1. Let p be an action (optionally non-free) of a finite group G on an aspherical space T . The diagonal free action p_f is induced on the space $T \times \mathbf{E}G$ where $\mathbf{E}G$ is a contractible space and G acts regularly and freely on it. The following extension is called a subordinate extension to the action p :

$$1 \rightarrow \pi_1(T) \rightarrow \pi_1(T \times \mathbf{E}G/G) \rightarrow G \rightarrow 1. \quad (3)$$

Now we show that the extension (3) is well defined. Thus we have to prove that this extension is independent of the choice of the space $\mathbf{E}G$. Let $\mathbf{E}G$ and $\mathbf{E}G'$ be contractible spaces and let G act regularly and freely on them. Since $\mathbf{E}G$ and $\mathbf{E}G'$ are contractible, then from theorem 3.1 it follows that the actions p_f and p'_f are strongly homotopy conjugate. Consequently there exists an equivariant homotopy equivalence $\psi : \mathbf{E}G \rightarrow \mathbf{E}G'$. By φ denote the map $\varphi : T \times \mathbf{E}G \rightarrow T \times \mathbf{E}G'$, $\varphi(x, e) = (x, \psi(e))$. Obviously, φ is an equivariant homotopy equivalence. So from lemma 2.1 it follows that the induced extensions are isomorphic. Moreover, if we fix the natural isomorphisms $\pi_1(T \times \mathbf{E}G) = \pi_1(T)$ and $\pi_1(T \times \mathbf{E}G') = \pi_1(T)$, then these extensions will be congruent.

Lemma 4.1. *If the action p is free, then the subordinate extension (3) and the extension from the classification theorem 2.1 are congruent.*

Proof. It is sufficient to prove that the actions p and p_f are strongly homotopy conjugate. The projection $T \times \mathbf{E}G \rightarrow T$ is the necessary equivariant homotopy equivalence. \square

It is easily shown that

Lemma 4.2. *Let p_1 and p_2 be an actions (optionally non-free) of finite groups G_1 and G_2 on aspherical spaces T_1 and T_2 . Suppose there exists a homotopy equivalence $\varphi : T_1 \rightarrow T_2$ and an isomorphism $\theta : G_1 \rightarrow G_2$ such that the equation (2) holds. Then the subordinate extensions are equivalent.*

Proof. From lemma 2.2 it follows that the actions of G_1 on $\mathbf{E}G_1$ and G_2 on $\mathbf{E}G_2$ are homotopy conjugate. Consequently there exists a homotopy equivalence $\Phi : \mathbf{E}G_1 \rightarrow \mathbf{E}G_2$ such that $\Phi(gx) = \theta(g)\Phi(x)$ for all $g \in G_1$ and $x \in T_1$.

Consider the homotopy equivalence $\varphi' : T_1 \times \mathbf{E}G_1 \rightarrow T_2 \times \mathbf{E}G_2$, $\varphi' : (x, e) \mapsto (\varphi(x), \Phi(e))$. Using lemma 2.1, we see that the extension induced by the diagonal actions of G_1 on $T_1 \times \mathbf{E}G_1$ and G_2 on $T_2 \times \mathbf{E}G_2$ are equivalent. \square

So the subordinate extension (3) allow us to distinguish actions of a finite group on aspherical spaces.

5 Calculation methods of subordinate extensions

Lemma 5.1 (about a fixed point). *If an action p of a finite group G on an aspherical space T has a fixed point, then the subordinate extension is decomposed, i.e., it is a semidirect product.*

Proof. There is a Cartan-Borel fibration $P : (T \times \mathbf{E}G)/G \rightarrow \mathbf{E}G/G$ associated with the action p where $P : G(x, e) \mapsto Ge$ for $x \in T$ and $e \in \mathbf{E}G$. The map P is induced by the equivariant projection $T \times \mathbf{E}G \rightarrow \mathbf{E}G$ and since G is finite, then P is locally trivial fibration. The fibre of this fibration is T .

It is clear that the exact sequence of this fibration

$$1 \rightarrow \pi_1(T) \rightarrow \pi_1((T \times \mathbf{E}G)/G) \xrightarrow{P_\#} \pi_1(\mathbf{E}G/G) \rightarrow 1 \quad (4)$$

is equivalent to the subordinate extension (3).

Let $x_0 \in T$ be a fixed point of the action p . By definition, put $S : \mathbf{E}G/G \rightarrow (T \times \mathbf{E}G)/G$. S is a section of P induced by the equivariant embedding $\mathbf{E}G \rightarrow T \times \mathbf{E}G$, $e \mapsto (x_0, e)$. Consider the maps $S_\#$ and $P_\#$ of the fundamental groups ($S_\#$ and $P_\#$ are induced by S and P). Since $P \circ S = id$, then the homomorphism $S_\#$ is a right inverse for $P_\#$. Q.E.D. \square

Note that the structure of the semidirect product on $\pi_1((T \times \mathbf{E}G)/G)$ is defined by the induced action of G on $\pi_1(T, x_0)$. This action is well defined because the point x_0 is fixed.

Also, relations between extensions subordinated to a group and its subgroup can help in some calculations.

Lemma 5.2. *Let a finite group G act on an aspherical space T and let H be its subgroup. Consider the induced action of H on T . Then the following diagram is commutative and has exact rows and columns⁴:*

$$\begin{array}{ccccccc}
 & 1 & & 1 & & 1 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 1 \longrightarrow & \pi_1(T) & \xrightarrow{P_\#^H} & \pi_1((T \times \mathbf{E}H)/H) & \xrightarrow{\partial} & H & \longrightarrow 1 \\
 & \downarrow id & & \downarrow P_\# & & \downarrow & \\
 1 \longrightarrow & \pi_1(T) & \xrightarrow{P_\#^G} & \pi_1((T \times \mathbf{E}G)/G) & \xrightarrow{\partial} & G & \longrightarrow 1 \\
 & \downarrow & & \downarrow j_1 & & \downarrow j_2 & \\
 & 1 & \longrightarrow & \pi_1((T \times \mathbf{E}G)/G) / \pi_1((T \times \mathbf{E}H)/H) & \xrightarrow{j_3} & G/H & \longrightarrow 1 \\
 & & & \downarrow & & \downarrow & \\
 & & & 1 & & 1 &
 \end{array}$$

⁴If H is not a normal subgroup of G , then the third row is considered as an exact sequence of pointed sets.

where $P^G : T \rightarrow T/G$, $P^H : T \rightarrow T/H$, and $P : T/H \rightarrow T/G$ are the natural projections.

Proof. Without loss of generality we can assume that the action p is free (the Borel construction allows us to instantly rewrite the following proof for the non-free case).

Consider the following commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{P^H} & T/H \\ & \searrow P^G \quad \nearrow P & \\ & T/G & \end{array}$$

Note that $P^G : T \rightarrow T/G$ and $P^H : T \rightarrow T/H$ are regular coverings with finite fibres. $P : T/H \rightarrow T/G$ is a covering and it is regular iff H is a normal subgroup of G .

Since the previous diagram is commutative, then the diagram from the conditions is commutative too. The first and the second rows are exact as they are parts of the long exact sequences of the coverings $P^H : T \rightarrow T/H$ and $P^G : T \rightarrow T/G$. The first and the third columns are obviously exact.

Let us show that the homomorphism $P_\#$ is a monomorphism: the map $H \rightarrow G$ is a monomorphism. Consequently, $\ker P_\# \subseteq \ker \partial = \text{Im } P_\#^H$. Therefore, $\ker P_\# \subseteq P_\#^H(\ker(P_\# \circ P_\#^H))$. On the other hand, $P_\# \circ P_\#^H = P_\#^G$ is a monomorphism. Thus, $\ker P_\# \subseteq P_\#^H(\ker(P_\#^G)) = \{e\}$. That is the second column is exact.

Finally, we shall prove that the last row is also exact. Let $j_1 : \pi_1(T/G) \rightarrow \pi_1(T/G)/\pi_1(T/H)$, $j_2 : G \rightarrow G/H$, and $j_3 : \pi_1(T/G)/\pi_1(T/H) \rightarrow G/H$ be the homomorphisms from the initial diagram. First we have to describe j_3 : suppose $x \in \pi_1(T/G)/\pi_1(T/H)$, then there exists $y \in \pi_1(T/G)$ such that $j_1(y) = x$ as j_1 is an epimorphism. Then we have $j_3(x) = (j_2 \circ \partial)(y)$. We see that $j_3(x)$ is independent of the choice of the element y because all elements from the coset $yP_\#(\pi_1(T/H))$ map to the same element:

$$(j_2 \circ \partial)(yP_\#(\pi_1(T/H))) = (j_2 \circ \partial)(y) \cdot (j_2 \circ \partial)(P_\#(\pi_1(T/H))) = (j_2 \circ \partial)(y).$$

1. Surjectiveness of j_3 : for any $x \in G/H$ there exists $y \in \pi_1(T/G)$ such that $(j_2 \circ \partial)(y) = x$ (j_2 and ∂ are epimorphisms). Hence $j_1(y) \in \pi_1(T/G)/\pi_1(T/H)$ is a preimage of x .
2. Injectiveness of j_3 : suppose $x \in \pi_1(T/G)/\pi_1(T/H)$ and $j_3(x) = 1$. Then there exists $y \in \pi_1(T/G)$ such that $j_1(y) = x$. Obviously, $\partial y \in \ker j_2 = H$. Hence there exists an element $z \in \pi_1(T/H)$ such that $\partial y = \partial z = \partial P_\#(z)$. Since $\partial(yP_\#(z^{-1})) = 1 \in G$; then there exists an element $t \in \pi_1(T)$ such that $P_\#^G(t) = yP_\#(z^{-1})$. Thus we get $P_\#(P_\#^H(t)z) = P_\#^G(t)P_\#(z) = y$, i.e., $y \in \text{Im } P_\#$. Consequently, $x = j_1(y) = 1$, Q.E.D.

□

Corollary 5.1. $\pi_1((T \times \mathbf{E}H)/H) = \partial^{-1}(H)$ is a subgroup in $\pi_1((T \times \mathbf{E}G)/G)$ and its index is the same as the index of H in G . Moreover, $\partial^{-1}(H)$ is a normal subgroup of $\pi_1((T \times \mathbf{E}G)/G)$ iff H is a normal subgroup of G .

Remark. If the action p of G on T is free and regular, then the induced action of the subgroup $H \subseteq G$ is also regular (the space $T/p(H)$ is a cell complex because the natural projection $T/p(H) \rightarrow T/p(G)$ is a covering and the space $T/p(G)$ is a cell complex).

The two previous lemmas are very useful because sometimes we can avoid explicit awkward calculations of the group $\pi_1((T \times \mathbf{E}G)/G)$.

Example. Since any connected graph is an aspherical space; then any action of a finite group on it induces a subordinate extension.

Let the dihedral group $G = D_{2n}$, $n \geq 3$ act on a circle $T = S^1$ (in this case the circle S^1 is an n -gon and the group D_{2n} acts on it by vertex permutation). The normal subgroup $\mathbb{Z}_n \subseteq D_{2n}$ acts freely and regularly on it and it is clear that $S^1/\mathbb{Z}_n = S^1$. Hence from lemma 5.2 it follows that the following diagram is commutative and its columns and rows are exact:

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \rightarrow & \mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z} & \rightarrow & \mathbb{Z}_n \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \rightarrow & \mathbb{Z} & \rightarrow & A & \rightarrow & D_{2n} \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & \rightarrow & \mathbb{Z}_2 & \rightarrow & \mathbb{Z}_2 \rightarrow 1 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 1 & & 1
 \end{array}$$

Thus the subgroup A is an extension of \mathbb{Z}_2 by \mathbb{Z} . Clearly, the only one variant is possible: $A = \mathbb{Z} \rtimes \mathbb{Z}_2$. So the extension subordinated to the natural action of D_{2n} on S^1 is

$$1 \rightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \rtimes_{(-1)} \mathbb{Z}_2 \rightarrow D_{2n} \rightarrow 1.$$

6 Spectral sequences

For any action p of a finite group G on a space T there exists a Cartan-Serre spectral sequence for the diagonal action on $T \times \mathbf{E}G$ (see [2]):

$$\begin{aligned}
 E_{p,q}^2 &= H_p(G, H_q(T)) \Rightarrow H_{p+q}((T \times \mathbf{E}G)/G); \\
 E_2^{p,q} &= H^p(G, H^q(T)) \Rightarrow H^{p+q}((T \times \mathbf{E}G)/G).
 \end{aligned} \tag{5}$$

Here the abelian groups $H_*(T)$ and $H^*(T)$ are G -modules induced by the action p .

Let T and $(T \times \mathbf{E}G)/G$ be cell complexes. So if the space T is aspherical, then $H^*(T)$ and $H^*((T \times \mathbf{E}G)/G)$ are a group cohomology and the spectral sequence (5) is the same as the spectral sequence of Hochschild-Mostov for the extension (3) subordinated by p :

$$\begin{aligned} E_{p,q}^2 &= H_p\left(G, H_q(\pi_1(T))\right) \Rightarrow H_{p+q}\left(\pi_1((T \times \mathbf{E}G)/G)\right); \\ E_2^{p,q} &= H^p\left(G, H^q(\pi_1(T))\right) \Rightarrow H^{p+q}\left(\pi_1((T \times \mathbf{E}G)/G)\right). \end{aligned}$$

Thus there is an information about relations between cohomology of the group G and the space T in the subordinated extension (3).

7 Group lattices

Theory of classifying spaces for small categories is well-known. It was developed in the last century by Segal and Quillen (see [9, 10]). Main ideas of these works previously appeared in Grothendieck's paper (see [7]). Some good results can be obtained on this field.

Let P be a poset. Consider P as a small category in natural sense. By $\mathbf{B}P$ denote the classifying space of the small category P . The space $\mathbf{B}P$ is a simplicial complex (with weak topology). Vertexes of $\mathbf{B}P$ are points in P , segments are inequalities $a < b$, triangles are inequalities $a < b < c$, etc.

Note that (co)homology $H_*(P)$ ($H^*(P)$), Eulerian characteristic, etc. of the poset P is exactly (co)homology $H_*(\mathbf{B}P)$ ($H^*(\mathbf{B}P)$), Eulerian characteristic, etc. of the classifying space $\mathbf{B}P$.

Let a finite group G act on P with respect to the order on P . Then this action naturally induce an action on the space $\mathbf{B}P$.

For any finite group G consider its lattices:

Definition 7.1. Let CG be the coset poset of G (without $\hat{0} = \emptyset$ and $\hat{1} = G$) with respect to the set-theoretic inclusion.

The poset of all cosets between g_1H_1 and g_2H_2 is called the segment between g_1H_1 and g_2H_2 and is denoted by $[g_1H_1, g_2H_2]$:

$$[g_1H_1, g_2H_2] = \{gH \in CG : g_1H_1 \subsetneq gH \subsetneq g_2H_2\}.$$

Without loss of generality we can consider only segments of the form $[H_1, H_2]$ where H_i is a subgroup of G or \emptyset (since any segment $[g_1H_1, g_2H_2]$ is isomorphic to the segment $[H_1, H_2]$).

Definition 7.2. The segment $[e, G]$ is called The subgroup lattice and is denoted by LG .

Remark. We exclude the maximal and the minimal elements from the lattices CG and LG because in the converse case the classifying spaces will be contractible and this is not informative.

Thus the group G acts on CG by conjugation and left shift with respect to the order. Also the action by conjugation induce an action on a segment $[N_1, N_2]$ including LG where N_1 and N_2 are normal subgroups of G .

Now consider the lattice CG (or LG) as a small category. An action of G on it gives us an action on the classifying space \mathbf{BCG} (or \mathbf{BLG}). So the proposed method of spectral sequences allows us to reveal a relation between the cohomology of the group G and the cohomology of the lattices CG and LG .

Theorem 7.1. *Let L be one of the lattices CG or LG and let an action of G on L (by conjugation or shift) be fixed. If the classifying space \mathbf{BL} is aspherical, then there exists a (co)homological spectral sequence convergent to the (co)homology of the group S :*

$$\begin{aligned} E_{p,q}^2 &= H_p(G, H_q(L)) \Rightarrow H_{p+q}(S), \\ E_2^{p,q} &= H^p(G, H^q(L)) \Rightarrow H^{p+q}(S), \end{aligned}$$

where $1 \rightarrow \pi_1(\mathbf{BL}) \rightarrow S \rightarrow G \rightarrow 1$ is the extension subordinated to the action of G on \mathbf{BL} .

Remark. If the lattice L is a wedge product of circles ($\pi_1(L)$ is a free non-abelian group) then G satisfies the conditions of the previous theorem. Subgroup lattices of many minimal simple groups are homotopy equivalent to wedges of circles. This fact was proved by Shareshian (see [11]). In this case the constructed spectral sequences are two-row.

Example. Let $G = \mathbb{Z}_{pq}$ where p and q are simple numbers and let $\Gamma_{pq} = \mathbf{BC}\mathbb{Z}_{pq}$ be the graph constructed by the lattice $C\mathbb{Z}_{pq}$. The shift action of G is considered. The following extension is subordinated to this action:

$$1 \rightarrow \pi_1(\Gamma_{pq}) \rightarrow S_{pq} \rightarrow \mathbb{Z}_{pq} \rightarrow 1.$$

There are $pq + p + q$ vertexes and $2pq$ edges in Γ_{pq} . Hence Γ_{pq} is homotopy equivalent to the wedge $\bigvee^{(p-1)(q-1)} S^1$ of $(p-1)(q-1)$ circles. Consequently, $H_0(\Gamma_{pq}) = \mathbb{Z}$, $H_1(\Gamma_{pq}) = \mathbb{Z}^{(p-1)(q-1)}$, $H_p(\Gamma_{pq}) = 0$ for all $p \geq 2$, and $\pi_1(\Gamma_{pq}) = F^{(p-1)(q-1)}$ is a non-abelian free group with $(p-1)(q-1)$ generators. Let us calculate the cohomology of the group S_{pq} by spectral sequence of Hochschild-Mostov: $H_k(\mathbb{Z}_{pq}, H_0(\Gamma_{pq})) = H_k(\mathbb{Z}_{pq})$ as \mathbb{Z}_{pq} acts trivially on $H_0(\Gamma_{pq}) = \mathbb{Z}$. Also it is easy to construct explicit generators of the group $H_1(\Gamma_{pq})$ in the graph Γ_{pq} , calculate the explicit action matrix of a generator of the group \mathbb{Z}_{pq} on $H_1(\mathbb{Z}_{pq})$, write out a necessary resolvent of \mathbb{Z}_{pq} for the module $H_1(\Gamma_{pq})$ by this matrix and get $H_k(\mathbb{Z}_{pq}, H_1(\Gamma_{pq})) = 0$ for all k ⁵. Thus we get $H_k(S_{pq}) = H_k(\mathbb{Z}_{pq})$ by spectral sequence.

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⁵The exact proof of this fact is removed from the paper texts because the used ideas are very simple and precise calculations are very cumbersome.

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